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From edge-disjoint paths to independent paths

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Abstract

Let $f(k)$ denote the maximum such that every simple undirected graph containing two vertices s, t and k edge-disjoint s–t paths, also contains two vertices u, v and $f(k)$ independent u–v paths. Here, a set of paths is *independent* if none of them contains an interior vertex of another. We prove that

$$f(k) = \begin{cases} k & \text{if } k \leq 2, \text{ and} \\ 3 & \text{otherwise.} \end{cases}$$

Since independent paths are edge-disjoint, it is clear that $f(k) \leq k$ for every positive integer k.

Let $\mathcal{P}$ be a set of edge-disjoint s–t paths in a graph G. Clearly, if $|\mathcal{P}| \leq 1$, then the paths in $\mathcal{P}$ are independent. If $\mathcal{P} = \{P_1, P_2\}$, a set of two independent u–v paths can easily be obtained as follows. Set $u := s$ and let v be the vertex that belongs to both P_1 and P_2 and is closest to s on P_1. Then, the u–v subpaths of P_1 and P_2 are independent. This proves that $f(k) = k$ if $k \leq 2$.

The lower bound for $f(k), k \geq 3$, is provided by the following lemma.

Lemma 1. *Let $G = (V, E)$ be a graph. If there are two vertices $s, t \in V$ with 3 edge-disjoint s–t paths in G, then there are two vertices $u, v \in V$ with 3 independent u–v paths in G.*

Proof. Let P_1, P_2, P_3 denote 3 edge-disjoint s–t paths, and let $S = \{s_1, s_2, s_3\}$, where s_i neighbors s on P_i, $1 \leq i \leq 3$. Consider the connected component G' of $G \setminus \{s\}$ containing t. Then, G' contains all vertices from S. Let T be a spanning tree of G'. Select v such that the s_i–v subpaths of T, $1 \leq i \leq 3$, are independent. This vertex v belongs to every subpath of T that has two vertices from S as endpoints. To see that this vertex exists, consider the s_1–s_3 subpath $P_{1,3}$ of T and the s_2–s_3 subpath $P_{2,3}$ of T. Set v to be the vertex that belongs to both $P_{1,3}$ and $P_{2,3}$ and is closest to s_2 on $P_{2,3}$ (if $P_{1,3}$ contains s_2, then $v = s_2$). Set $u := s$, and obtain 3 independent u–v paths in G by moving from u to s_i, and then along the s_i–v subpath of T to v, $1 \leq i \leq 3$. $\square$

For the upper bound, consider the following family of graphs, the *recursive diamond graphs* [4]. The recursive diamond graph of order 0 is $G_0 = (\{s, t\}, \{st\})$, and the diamond graph G_p of order $p \geq 1$ is obtained from G_{p-1} by replacing each edge $e = xy$ by the set of edges $\{xp_e, p_e y, xq_e, q_e y\}$, where p_e and q_e are new vertices. See Figure 1 for an illustration.

The following lemma entails the upper bound for $f(k), k \geq 3$.

Lemma 2. *For every $k \geq 3$, there is a graph $G = (V, E)$ containing two vertices $s, t \in V$ with k edge-disjoint s–t paths, but no two vertices $u, v \in V$ with 4 independent u–v paths.*

Proof. Consider the diamond graph $G = G_p$ of order $p = \lceil \log k \rceil$. G has $2^p \geq k$ edge-disjoint s–t paths. Let u, v be any two vertices in G. We will show that there are at most 3 independent u–v paths.

Observe that each recursive diamond graph G_r contains 4 edge-disjoint copies of G_{r-1}. The *extremities* of G_r are the vertices s and t, and the *extremities* of a subgraph H of G_r that is isomorphic to $G_{r'}, r' < r$, are the two vertices from H whose neighborhoods in G_r are not a subset of $V(H)$.

Let Q be the smallest vertex set containing u and v such that $G[Q]$ is a recursive diamond graph. Let q be the order of the recursive diamond graph $G[Q]$.

If $q = 0$, then uv is an edge in G, and either u or v has degree 2. But then, the number of independent u–v paths in G is at most 2 since independent paths pass through distinct neighbors of u and v.

If $q > 0$, then uv is not an edge in G. Decompose $G[Q]$ into 4 edge-disjoint graphs $H_1, \ldots, H_4$ isomorphic to G_{q-1} such that $u \in V(H_1)$ and the H_i are ordered cyclically by their index (i.e., $V(H_1) \cap$

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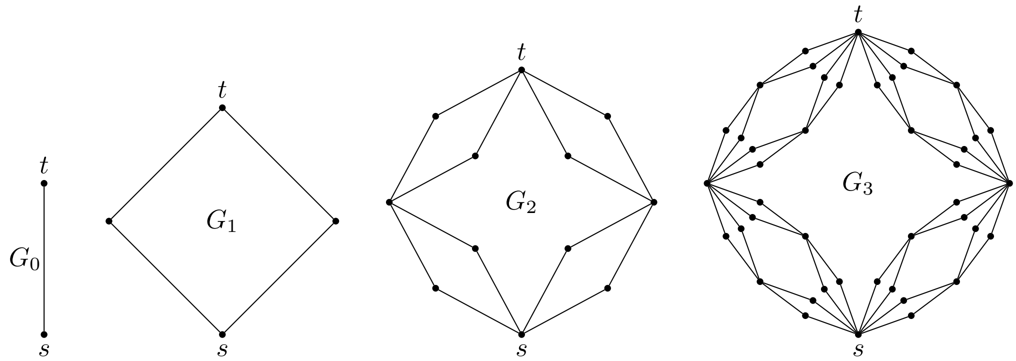

Figure 1: The recursive diamond graphs of order $0, 1, 2$, and 3.

$V(H_3) = \emptyset$). Since we chose Q to be minimum, u and v do not belong to the same $H_i, 1 \leq i \leq 4$. If $u \notin V(H_2) \cup V(H_4)$, then the extremities of H_1 are a u–v-vertex cut of size 2 in $G[Q]$ and in G. Otherwise, suppose, without loss of generality, that $u \in V(H_1) \cap V(H_2)$. Since $v \notin V(H_1) \cup V(H_2)$, the other two extremities of H_1 and H_2 form a u–v-vertex cut C of size 2 in $G[Q]$. The set C is also a u–v-vertex cut in G, unless $q < p$ and u is an extremity of another subgraph J of G isomorphic to G_q that is edge-disjoint from $G[Q]$. In the latter case, add the other extremity of J to C to obtain a u–v-vertex cut in G of size 3.

Since G has a u–v-vertex cut of size at most 3, by Menger's theorem [6], there are at most 3 independent u–v paths in G. □

An application Lemma 1 has been used in an algorithm [2] for the detection of backdoor sets to ease Satisfiability solving. A backdoor set of a propositional formula is a set of variables such that assigning truth values to the variables in the backdoor set moves the formula into a polynomial-time decidable class; see [3] for a survey. The class of nested formulas was introduced by Knuth [5] and their satisfiability can be decided in polynomial time. To find a backdoor set to the class of nested formulas, the algorithm from [2] considers the clause-variable incidence graph of the formula. If the formula is nested, this graph does not contain a $K_{2,3}$-minor with the additional property that the independent set of size 3 is obtained by contracting 3 connected subgraphs containing a variable each. In the correctness proof of the algorithm it is shown that in certain cases the formula does not have a small backdoor set. This is shown by exhibiting two vertices u, v and 3 independent u–v paths in an auxiliary graph using Lemma 1. Expanding these edges to the paths they represent in the formula's incident graph gives rise to a $K_{2,3}$-minor with the desired property.

On the other hand, Lemma 2 shows the limitations of this approach if we would like to enlarge the target class to more general formulas.

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